THE FIRST EIGENVALUE OF DIRAC AND LAPLACE OPERATORS ON SURFACES

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Abstract. Let (M, g, σ) be a compact Riemmannian surface equipped with a spin structure σ. For any metric $\tilde{g}$ on M, we denote by $\mu_1(\tilde{g})$ (resp. $\lambda_1(\tilde{g})$) the first positive eigenvalue of the Laplacian (resp. the Dirac operator) with respect to the metric $\tilde{g}$. In this paper, we show that

$$\inf \frac{\lambda_1(\tilde{g})^2}{\mu_1(\tilde{g})} \leqslant \frac{1}{2}.$$

where the infimum is taken over the metrics $\tilde{g}$ conformal to g. This answer a question asked by Agricola, Ammann and Friedrich in [AAF99].

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Contents

1. Introduction

Let (M, g, σ) be a compact Riemannian surface equipped with a spin structure σ. For any metric $\bar{g}$ on M, we denote by $\Sigma_{\bar{g}}M$ the spinor bundle associated to $\bar{g}$. We let $\Delta_{\bar{g}}$ be the Laplace-Beltrami operator acting on smooth functions of M and $D_{\bar{g}}$ be the Dirac operator acting on smooth spinor fields with respect to the metric $\bar{g}$. We also denote by $\mu_1(\bar{g})$ (resp. $\lambda_1(\bar{g})$) the smallest positive eigenvalue of $\Delta_{\bar{g}}$ (resp. $D_{\bar{g}}$). Agricola, Ammann and Friedrich asked the following question in [AAF99]:

When M is a two dimensional torus, can we find on M a Riemannian metric $\tilde{g}$ for which $\lambda_1(\tilde{g})^2 < \mu_1(\tilde{g})$?

The main goal of this article is to answer this question. We prove the

Theorem 1.1. *There exists a family of metrics $(g_\varepsilon)_\varepsilon$ conformal to g for which*

$$\limsup_{\varepsilon\to 0} \lambda_1(g_\varepsilon)^2 \mathrm{Vol}_{g_\varepsilon}(M) \leqslant 4\pi$$

$$\liminf_{\varepsilon\to 0}\ \mu_1(g_\varepsilon)\mathrm{Vol}_{g_\varepsilon}(M) \geqslant 8\pi.$$

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Theorem 1.1 clearly answers the question of [AAF99] but says much more: first, the result is true on any compact Riemannian surface equipped with a spin structure and not only when M is a two-dimensional torus. In addition, the metric $\tilde{g}$ can be chosen in a given conformal class. Finally, this metric $\tilde{g}$ can be chosen such that $(2-\delta)\lambda_1(g)^2 < \mu_1(g)$ where $\delta > 0$ is arbitrary small. More precisely Theorem 1.1 shows

Corollary 1.2. *On any compact Riemannian surface (M, g), we have*

$$\inf \frac{\lambda_1(\bar{g})^2}{\mu_1(\bar{g})} \leqslant \frac{1}{2}$$

where the infimum is taken over the metric $\bar{g}$ conformal to g.

Theorem 1.1 has other interesting consequences. Indeed, it proves

Corollary 1.3. *For any compact surface (M, g) equipped with a spin structure σ, we let*

$$\lambda^+_{\min}(M, g, \sigma) = \inf \lambda_1(\bar{g}) \mathrm{Vol}_{\bar{g}}^{\frac{1}{2}}(M)$$

where the infimum is taken over the metrics $\bar{g}$ conformal to g. Then, we have $\lambda^+_{\min}(M, g, \sigma) \leqslant \lambda^+_{\min}(\mathbb{S}^2)$ where $\lambda^+_{\min}(\mathbb{S}^2)$ is the same invariant computed on the standard sphere $\mathbb{S}^2$.

This corollary is an immediate consequence of the fact that $\lambda^+_{\min}(\mathbb{S}^2) = 2\sqrt{\pi}$ (see [AHM03]). This result was announced in [AHM03]. The conformal invariant $\lambda^+_{\min}$ has been studied in many papers (see for example [Hij86, Lot86, Bär92, Amm03, AHM03, AH06]). Indeed, it has many relations with Yamabe problem (see [LP87]). Corollary 1.3 has been proved in all dimensions by Ammann in [Amm03] if either $n \geqslant 3$ or is D is invertible. Corollary 1.3 extends the result to the remaining case: $n = 2$ and $Ker(D) \neq \{0\}$. In [AHM03], an alternative proof of the case $n \geq 3$ is given and the proof of the case $n = 2$ is skectched.

In the same spirit, a consequence of Theorem 1.1 is

Corollary 1.4. *For any compact surface (M, g), we let*

$$\mu_{\sup}(M, g) = \sup \mu_1(\bar{g}) \mathrm{Vol}_{\bar{g}}^{\frac{1}{2}}(M)$$

where the infimum is taken over the metrics $\bar{g}$ conformal to g. Then, we have $\mu_{\sup}(M, g) \geqslant \mu_{\sup}(\mathbb{S}^2)$ where $\mu_{\sup}(\mathbb{S}^2)$ is the same invariant computed on the standard sphere $\mathbb{S}^2$.

The invariant $\mu_{\sup}$ has been studied in [CoES03] and Corollary 1.4 is a particular case of Theorem A in this paper. We obtain here another proof.

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2. Generalized metrics

Let f be a smooth positive function and set $\bar{g} = f^2 g$. Let also for $u \in C^\infty(M)$

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_{\bar{g}} dv_{\bar{g}}}{\int_M u^2 dv_{\bar{g}}}.$$

It is well known that $\mu_1(\bar{g}) = \inf I_{\bar{g}}(u)$ where the infimum is taken over the smooth non-zero functions u for which $\int_M u dv_{\bar{g}} = 0$. We now can write all these expressions in the metric g. We then see that for $u \in C^\infty(M)$, we have

$$I_{\bar{g}}(u) = \frac{\int_M |\nabla u|_g^2 dv_g}{\int u^2 f^2 dv_g}$$

and $\mu_1(\bar{g}) = \inf I_{\bar{g}}(u)$ where the infimum is taken over the smooth non-zero functions u for which $\int_M u f^2 dv_g = 0$. Now if f is only of class $C^{0,a}(M)$ for some $a > 0$, we can define $\bar{g} = f^2 g$. The 2-form $\bar{g}$ is not really a metric since f is not smooth. We then say that g is a *generalized metric*. We can

define the first eigenvalue $\mu_1(\bar{g})$ of $\Delta_{\bar{g}}$ using the definition above. Now, by standard methods, one sees that there exists a function $u \in C^2(M)$ with $\int_M uf^2 dv_g = 0$ and such that $I_{\bar{g}}(u) = \mu_1(\bar{g})$. Writing the Euler equation for u, we see that

$$\Delta_g u = \mu_1(\bar{g}) f^2 u. \tag{1}$$

We prove the following result

Lemma 2.1. *If (f_n) is a sequence of smooth positive functions which converges uniformily to f, then $\mu_1(f_n^2 g)$ tends to $\mu_1(\bar{g})$.*

Proof. Let u_n be a eigenfunction function associated to $\mu_1(f_n^2 g)$. Without loss of generality, we can assume that $\displaystyle\int_M u_n^2 f_n^2 = 1$. We set $v_n = u_n - \dfrac{\displaystyle\int_M u_n f^2 dv_g}{\displaystyle\int_M f^2 dv_g}$. We then have $\displaystyle\int_M v_n f^2 dv_g = 0$ and hence

$$\mu_1(\bar{g}) \leqslant I_{\bar{g}}(v_n). \tag{2}$$

We have

$$\int_M |\nabla v_n|^2 dv_g = \int_M |\nabla u_n|^2 dv_g = \int_M u_n \Delta_g u_n dv_g.$$

By equation (1), we get that

$$\int_M |\nabla v_n|^2 dv_g = \mu_1(f_n^2 g) \int_M f_n^2 u_n^2 dv_g = \mu_1(f_n^2 g). \tag{3}$$

We also have

$$\int_M f^2 v_n^2 dv_g = \int_M f^2 u_n^2 - \frac{\left(\displaystyle\int_M u_n f^2 dv_g\right)^2}{\displaystyle\int_M f^2 dv_g}.$$

Now,

$$\left|\int_M u_n f^2 dv_g\right| = \left|\int_M u_n (f^2 - f_n^2) dv_g\right| \leqslant C \int_M |u_n| (f + f_n)^2 \parallel f - f_n \parallel_\infty .$$

Since the sequence $(f_n)_n$ tends uniformly to f and since $\displaystyle\int_M f_n^2 u_n^2 dv_g = 1$, we get that $\lim_n \displaystyle\int_M u_n f^2 dv_g = 0$. In the same way,

$$\int_M f^2 u_n^2 dv_g = \int_M f_n^2 u_n^2 dv_g + o(1) = 1 + o(1).$$

Finally, we obtain

$$\int_M f^2 v_n^2 dv_g = 1 + o(1).$$

Together with (2) and (3), we obtain that $\mu_1(\bar{g}) \leqslant \liminf_n \mu_1(f_n^2 g)$. Now, let u be associated to $\mu_1(\bar{g})$ and set $v = u - \dfrac{\displaystyle\int_M u f_n^2 dv_g}{\displaystyle\int_M f_n dv_g}$. We have $\displaystyle\int_M v^2 f_n^2 dv_g = 0$ and hence

$$\mu_1(f_n^2 g) \leqslant I_{f_n^2 g}(v).$$

It is easy to see that $\lim_n I_{f_n^2 g}(v) = I_{\bar{g}}(u) = \mu_1(\bar{g})$. We then obtain that $\mu_1(\bar{g}) \geqslant \limsup_n \mu_1(f_n^2 g)$. This proves Lemma 2.1. □

In the same way, if $\bar{g} = f^2 g$ is a metric conformal to g where f is positive and smooth, we define

$$J'_{\bar{g}}(\psi) = \frac{\displaystyle\int_M |D_{\bar{g}}\psi|^2_{\bar{g}} f^{-1} dv_{\bar{g}}}{\displaystyle\int_M \langle D_{\bar{g}}\psi, \psi\rangle_{\bar{g}} dv_{\bar{g}}}.$$

The first eigenvalue of the Dirac operator $D_{\bar{g}}$ is then given by $\lambda_1^+(\bar{g}) = \inf J'_{\bar{g}}(\psi)$ where the infimum is taken over the smooth spinor fields ψ for which $\int_M \langle D_g\psi, \psi\rangle dv_g > 0$. Now, it is well known (see [Hit74, Hij01]) that where can identify isometrically on each fiber spinor fields for the metric g and spinor fields for the metric $\bar{g}$. Moreover, we have for all smooth spinor field:

$$D_{\bar{g}}(f^{-\frac{1}{2}}\psi) = f^{-\frac{3}{2}} D_g\psi.$$

This implies that if we set $\varphi = f^{-\frac{1}{2}}\psi$, we have

$$J_{\bar{g}}(\varphi) := \frac{\displaystyle\int_M |D_g\varphi|^2 f^{-1} dv_g}{\displaystyle\int_M \langle D_g\varphi, \varphi\rangle dv_g} = J'_{\bar{g}}(\psi)$$

and the first eigenvalue of the Dirac operator $D_{\bar{g}}$ is given by $\lambda_1^+(\bar{g}) = \inf J_{\bar{g}}(\varphi)$ where the infimum is taken over the smooth spinor fields φ for which $\displaystyle\int_M \langle D\varphi, \varphi\rangle dv_g > 0$. With the definition above, we can extend the definition of $\lambda_1(\bar{g})$ when $\bar{g}$ is a generalised metric. By standard methods, there exists a spinor fields $\varphi \in C^1(M)$ such that $\lambda_1^+(\bar{g}) = J_{\bar{g}}(\varphi)$ and such that

$$D_g\varphi = \lambda_1^+(\bar{g}) f\varphi. \tag{4}$$

We then have a result similar to Lemma 2.1:

Lemma 2.2. *If (f_n) is a sequence of smooth positive functions which converges uniformly to f, then $\lambda_1(f_n^2 g)$ tends to $\lambda_1(\bar{g})$.*

The proof is similar to the one of Lemma 2.1 and we omit it here.

3. The metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$

In this paragraph, we construct the metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ which will satisfy:

$$\limsup_{\varepsilon\to 0} \lambda_1(g_{\alpha,\varepsilon})^2 \mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) \leqslant 4\pi + C(\alpha) \tag{5}$$

where $C(\alpha)$ is a positive constant which goes to 0 with α and

$$\liminf_{\varepsilon\to 0} \mu_1(g_{\alpha,\varepsilon}) \mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) \geqslant 8\pi. \tag{6}$$

Clearly this implies Theorem 1.1. By Lemmas 2.1 and 2.2, one can assume that the metrics $(g_{\alpha,\varepsilon})_{\alpha,\varepsilon}$ are generalized metrics. We just have to define the volume of M for generalized metric by $\mathrm{Vol}_{f^2 g}(M) = \int_M f^2 dv_g$. At first, without loss of generality, we can assume that g is flat near a point $p \in M$. Let $\alpha > 0$ be a small number to be fixed later such that g is flat on $B_p(\alpha)$. We set for all $x \in M$ and $\varepsilon > 0$,

$$f_{\alpha,\varepsilon}(x) = \begin{cases} \frac{\varepsilon^2}{\varepsilon^2 + r^2} & \text{if} \quad r \leqslant \alpha \\ \frac{\varepsilon^2}{\varepsilon^2 + \alpha^2} & \text{if} \quad r > \alpha \end{cases}$$

where $r = d_g(.,p)$. The function $f_{\alpha,\varepsilon}$ is of class $C^{0,a}$ for all $a \in]0,1[$ and is positive on M. We then define for all $\varepsilon > 0$, $g_{\alpha,\varepsilon} = f_{\alpha,\varepsilon}^2 g$. The 2-forms $(g_{\alpha,\varepsilon})_{\alpha,ep}$ will be the desired generalized metrics. For these metrics, we have

$$\mathrm{Vol}_{g_{\alpha,\varepsilon}}(M) = \int_M f_{\alpha,\varepsilon}^2 dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g + \int_{M\setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g.$$

Since g is flat on $B_p(\alpha)$, we have

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = \int_0^{2\pi}\int_0^{\alpha} \frac{\varepsilon^4 r}{(\varepsilon^2+r^2)^2} dr d\Theta.$$

Setting $y = \frac{x}{\varepsilon}$ we obtain:

$$\int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = 2\pi\varepsilon^2 \int_0^{\frac{\alpha}{\varepsilon}} \frac{r}{(1+r^2)^2} dr = 2\pi\varepsilon^2 \left(\int_0^{+\infty} \frac{r}{(1+r^2)^2} dr + o(1) \right) = \pi\varepsilon^2 + o(\varepsilon^2).$$

Since $f_{\alpha,\varepsilon}^2 \leqslant \frac{\varepsilon^4}{\alpha^4}$ on $M \setminus B_p(\alpha)$, we have $\int_{M\setminus B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g = o(\varepsilon^2)$. We obtain

$$\mathrm{Vol}_{g_\varepsilon}(M) = \pi\varepsilon^2 + o(\varepsilon^2). \tag{7}$$

In the whole paper, the notation "$o(.)$" must be understood as ε tends to 0.

4. Proof of relation (5)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = \frac{2}{1+|x|^2}$. Let ψ_0 be a non-zero parallel spinor field on $\mathbb{R}^2$ such that $|\psi_0|^2 = 1$. As in [AHM03], we set on $\mathbb{R}^2$

$$\psi(x) = f^{\frac{n}{2}}(x)(1-x)\cdot\psi_0.$$

As easily computed, we have on $\mathbb{R}^2$

$$D\psi = f\psi \ \text{ and } \ |\psi| = f^{\frac{1}{2}}. \tag{8}$$

Now, we fixe a small number $\delta > 0$ such that g is flat on $B_p(\delta)$. Then, we take $\varepsilon \leqslant \alpha \leqslant \delta$. We will let ε goes to 0. We let also η be a smooth cut-off function defined on M such that $0 \leqslant \eta \leqslant 1$, $\eta(B_p(\delta)) = \{1\}$, $\eta(M \setminus B_p(2\delta)) = \{0\}$. Identifying $B_p(\delta)$ in M with $B_0(\delta)$ in $\mathbb{R}^2$, we can define a smooth spinor field on M by $\psi_\varepsilon = \eta(x)\psi\left(\frac{x}{\varepsilon}\right)$. Using (8), we have

$$D_g(\psi_\varepsilon) = \nabla\eta\cdot\psi\left(\frac{x}{\varepsilon}\right) + \frac{\eta}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)\psi\left(\frac{x}{\varepsilon}\right). \tag{9}$$

Since $\langle \nabla\eta\cdot\psi(\frac{x}{\varepsilon}), \psi(\frac{x}{\varepsilon})\rangle \in i\mathbb{R}$ and since $|D_g\psi_\varepsilon|^2 \in \mathbb{R}$, we have

$$\int_M |D_g\psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1} dv_g = I_1 + I_2 \tag{10}$$

where

$$I_1 = \int_M |\nabla\eta|^2 \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 dx \ \text{ and } I_2 = \int_M \frac{\eta^2}{\varepsilon^2} f^2\left(\frac{x}{\varepsilon}\right)\left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 f_{\alpha,\varepsilon}^{-1} dx.$$

At first, let us deal with I_1. By (8),

$$I_1 \leqslant C\int_M f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx = C\int_{B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx + C\int_{B_p(2\delta)\setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx$$

where, as in the following, C denotes a constant independant of α and ε. On $B_p(\alpha)$, $f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} = 2$. Hence,

$$\int_{B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} dx \leqslant C\alpha^2.$$

On $B_p(2\delta) \setminus B_p(\alpha)$, since $\varepsilon \leqslant \alpha$,

$$f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1} \leqslant \frac{4\alpha^2}{\varepsilon^2+r^2} = \frac{4\alpha^2}{\varepsilon^2(1+\left(\frac{r}{\varepsilon}\right)^2)}$$

Hence,

$$\begin{aligned}\int_{B_p(2\delta)\setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1}dx &\leqslant \frac{4\alpha^2}{\varepsilon^2}\int_0^{2\pi}\int_\alpha^\delta \frac{r}{\left(1+\left(\frac{r}{\varepsilon}\right)^2\right)}drd\Theta \\ &\leqslant 8\pi\alpha^2\int_{\frac{\alpha}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \frac{r}{(1+r^2)}dr \\ &\leqslant 8\pi\alpha^2\ln\left(\frac{\varepsilon^2+\delta^2}{\varepsilon^2+\alpha^2}\right).\end{aligned}$$

We get

$$\int_{B_p(2\delta)\setminus B_p(\alpha)} f\left(\frac{x}{\varepsilon}\right) f_{\alpha,\varepsilon}^{-1}dx \leqslant C\alpha^2\ln\left(\frac{2\delta^2}{\alpha^2}\right).$$

Finally, we obtain

$$I_1 \leqslant C\alpha^2 + C\ln\left(\frac{2\delta^2}{\alpha^2}\right) = a(\alpha) \tag{11}$$

where $a(\alpha)$ goes to 0 with α. Now, by (8),

$$I_2 \leqslant C\int_{B_p(2\delta)} f^3\left(\frac{x}{\varepsilon}\right)f_{\alpha,\varepsilon}^{-1}dx.$$

Since $f_{\alpha,\varepsilon} \geqslant \frac{1}{2}f(\frac{x}{\varepsilon})$, we have

$$I_2 \leqslant \frac{2}{\varepsilon^2}\int_{B_p(2\delta)} f^2\left(\frac{x}{\varepsilon}\right)dx.$$

Mimicking what we did to get (7), we obtain that

$$I_2 \leqslant 8\pi + o(1)$$

when ε tends to 0. Together with (10) and (11), we obtain

$$\int_M |D_g\psi_\varepsilon|^2 f_{\alpha,\varepsilon}^{-1}dv_g \leqslant 8\pi + a(\alpha) + o(1). \tag{12}$$

In the same way, by (9), since $\int_M \langle D_g(\psi_\varepsilon),\psi_\varepsilon\rangle dv_g \in \mathbb{R}$ and since $\langle \nabla\eta\cdot\psi(\frac{x}{\varepsilon}),\psi(\frac{x}{\varepsilon})\rangle \in i\mathbb{R}$, we have

$$\int_M \langle D_g(\psi_\varepsilon),\psi_\varepsilon\rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)\left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 dv_g.$$

By (8), this gives

$$\int_M \langle D_g(\psi_\varepsilon),\psi_\varepsilon\rangle dv_g = \int_M \frac{\eta^2}{\varepsilon} f^2\left(\frac{x}{\varepsilon}\right) dv_g.$$

With the computations made above, it follows that

$$\int_M \langle D_g(\psi_\varepsilon),\psi_\varepsilon\rangle dv_g = 4\pi\varepsilon + o(\varepsilon).$$

Together with (12) and (7), we obtain

$$\lambda_1(g_{\alpha,\psi})^2\mathrm{Vol}_{g_{\alpha,\psi}}(M) \leqslant \left(J_{g_{\alpha,\psi}}(\psi_\varepsilon)\right)^2\mathrm{Vol}_{g_{\alpha,\psi}}(M) \leqslant \left(\frac{8\pi+a(\alpha)+o(1)}{4\pi\varepsilon+o(\varepsilon)}\right)^2(\pi\varepsilon^2+o(\varepsilon^2)) = \frac{1}{\varepsilon}\left(4\pi+a(\alpha)+o(1)\right).$$

Relation (5) immediatly follows.

5. Proof of relation (6)

First we need the following estimate

Lemma 5.1. *For any $\varepsilon > 0$ and $u \in C_c^\infty(B_p(\alpha))$, then*

$$\int_M u^2 f_{\alpha,\varepsilon}^2 dv_g \leqslant \frac{\varepsilon^2}{8}\int_M |\nabla u|^2 dv_g + \frac{1}{\pi\varepsilon^2}\left(\int_M u f_{\alpha,\varepsilon}^2 dv_g\right)^2.$$

Proof. Let $g_\varepsilon = f_{\alpha,\varepsilon}^2 g$. Then $(B_p(\alpha), g_\varepsilon)$ is embedded in a canonical sphere of volume $\displaystyle\int_{\mathbb{R}^2}\left(\frac{\varepsilon^2}{\varepsilon^2+r^2}\right)^2 dx = \pi\varepsilon^2$. Then from the Poincaré-Sobolev inequality, we have

$$\int_M u^2 dv_{g_\varepsilon} \leqslant \frac{1}{\mu_{1,\varepsilon}}\int_M |\nabla^\varepsilon u|_{g_\varepsilon}^2 dv_{g_\varepsilon} + \frac{1}{V_\varepsilon}\left(\int_M u dv_{g_\varepsilon}\right)^2$$

where $\mu_{1,\varepsilon} = \frac{8}{\varepsilon^2}$ is the first nonzero eigenvalue of the Laplacian on the sphere of volume $V_\varepsilon = \pi\varepsilon^2$ and $\nabla^\varepsilon u$ denotes the gradient of u with respect to the metric g_ε. Now since $|\nabla^\varepsilon u|_{g_\varepsilon}^2 = f_{\alpha,\varepsilon}^{-2}|\nabla u|_g^2$ and $dv_{g_\varepsilon} = f_{\alpha,\varepsilon}^2 dv_g$, we get the desired result.

□

Lemma 5.2. *For any $u, v \in C^\infty(M)$, we have*

$$\int_M (\Delta u) u v^2 dv_g = \int_M |\nabla(uv)|_g^2 dv_g - \int_M u^2 |\nabla v|_g^2 dv_g.$$

Proof. The proof is an elementary calculation.

□

Because of the relation (7), the inequality (6) is equivalent to the following

$$\liminf_{\varepsilon \longrightarrow 0} \varepsilon^2 \mu_1(g_\varepsilon) \geqslant .8 \tag{13}$$

In order to prove this inequality, we assume that for any ε small enough, there exists k, $0 < k < 1$ so that

$$\mu_1(g_\varepsilon) < \frac{8}{\varepsilon^2}k. \tag{14}$$

Let u_ε be an eigenfunction associated to $\mu_1(g_\varepsilon)$. Then $u_\varepsilon \in C^2(M)$ and $\Delta_{g_\varepsilon} u_\varepsilon = \mu_1(g_\varepsilon) u_\varepsilon$ where Δ_{g_ε} denotes the Laplacian associated to the metric g_ε. Since the dimension is 2, $\Delta_{g_\varepsilon} = \frac{1}{f_{\alpha,\varepsilon}^2}\Delta$ and

$$\Delta u_\varepsilon = \mu_1(g_\varepsilon) f_{\alpha,\varepsilon}^2 u_\varepsilon. \tag{15}$$

We normalize u_ε so that $\|u_\varepsilon\|_{H_1^2} = 1$. Up to a subsequence we can assume that $\displaystyle\int_M |\nabla u_\varepsilon|^2 dv_g \longrightarrow l$ and $\displaystyle\int_M u_\varepsilon^2 dv_g \longrightarrow l'$ with $l + l' = 1$. Since (u_ε) is bounded in H_1^2, there exists a subsequence so that $u_\varepsilon \longrightarrow u$ weakly in H_1^2. In the following, all the convergences are up to subsequence. We sometimes omit to recall this fact.

Lemma 5.3. *There exists a constant c_0 such that $u = c_0$.*

Proof. Let $\varphi \in C^\infty(M)$ and

$$\eta_\rho := \begin{cases} 1 & \text{on} \quad B_p(\rho) \\ 0 & \text{on} \quad M \setminus B_p(2\rho) \end{cases}$$

satisfying $0 \leqslant \eta_\rho \leqslant 1$ and $|\nabla \eta_\rho| \leqslant \frac{1}{\rho}$. We have

$$\int_M \langle \nabla u, \nabla \varphi \rangle = \int_M \langle \nabla u, \nabla(\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla u, \nabla((1-\eta_\rho)\varphi) \rangle dv_g. \tag{16}$$

Now we have

$$\begin{aligned} \int_M \langle \nabla u, \nabla(\eta_\rho \varphi) \rangle dv_g &= \int_M \langle \nabla u, \nabla \eta_\rho \rangle \varphi dv_g + \int_M \langle \nabla u, \nabla \varphi \rangle \eta_\rho dv_g \\ &\leqslant C \left(\int_{B_p(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left(\int_{B_p(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2} \\ &+ \left(\int_{B_p(2\rho)} |\nabla u|^2 dv_g \right)^{1/2} \left(\int_{B_p(2\rho)} |\nabla \varphi|^2 dv_g \right)^{1/2}. \end{aligned}$$

The limit of the last term is 0 when $\rho \longrightarrow 0$. Moreover from the definition of η_ρ and from the fact that M is a 2-dimensional locally flat domain, the limit of $\left(\int_{B_p(2\rho)} |\nabla \eta_\rho|^2 dv_g \right)^{1/2}$ is bounded in a neighborhood of 0. Then we deduce that

$$\int_M \langle \nabla u, \nabla(\eta_\rho \varphi) \rangle dv_g \longrightarrow 0 \tag{17}$$

when $\rho \longrightarrow 0$. On the other hand

$$\begin{aligned} \left| \int_M \langle \nabla u, \nabla\left((1-\eta_\rho)\varphi\right) \rangle dv_g \right| &= \lim_{\varepsilon \longrightarrow 0} \left| \int_M \langle \nabla u_\varepsilon, \nabla\left((1-\eta_\rho)\varphi\right) \rangle dv_g \right| \\ &= \lim_{\varepsilon \longrightarrow 0} \left| \int_M (\Delta u_\varepsilon)(1-\eta_\rho)\varphi dv_g \right| \\ &= \lim_{\varepsilon \longrightarrow 0} \left| \mu_1(g_\varepsilon) \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (1-\eta_\rho)\varphi dv_g \right|. \end{aligned}$$

Now from the definition of $f_{\alpha,\varepsilon}$ and from (14) we get

$$\left| \mu_1(g_\varepsilon) \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (1-\eta_\rho)\varphi dv_g \right| \leqslant \frac{8}{\varepsilon^2} k C \varepsilon^4 \left(\int_M u_\varepsilon^2 dv_g \right)^{1/2} \left(\int_M (1-\eta_\rho)\varphi^2 dv_g \right)^{1/2}$$

where C is a constant depending on the compact support of $(1-\eta_\rho)\varphi$. Then making $\varepsilon \longrightarrow 0$, we deduce that

$$\int_M \langle \nabla u, \nabla\left((1-\eta_\rho)\varphi\right) \rangle dv_g = 0.$$

Now, reporting this and (17) in (16) we obtain that $\int_M \langle \nabla u, \nabla \varphi \rangle dv_g = 0$ and $\Delta u = 0$ on M in the sense of distributions. This implies that $u \equiv c_0$ on M for a constant c_0.

□

Lemma 5.4. *Let $(c_\varepsilon)_\varepsilon$ be a bounded sequence of real numbers. Then*

$$\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g \leqslant O(\varepsilon^2 \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \varepsilon^4).$$

Proof. Let η be a C^∞ function defined on M so that

$$\eta := \begin{cases} 1 & \text{on} \quad B_p(\alpha/2) \\ 0 & \text{on} \quad M \setminus B_p(\alpha) \end{cases}$$

satisfying $0 \leqslant \eta \leqslant 1$ and $|\nabla \eta| \leqslant 1$.
From the lemma 5.1, we have

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant \frac{\varepsilon^2}{8} \int_M |\nabla((u_\varepsilon - c_\varepsilon)\eta)|^2 dv_g + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g \right)^2$$

and applying the lemma 5.2 to the first term of the right hand side, we get

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant$$
$$\frac{\varepsilon^2}{8} \int_M (\Delta(u_\varepsilon - c_\varepsilon))(u_\varepsilon - c_\varepsilon)\eta^2 dv_g + \frac{\varepsilon^2}{8} \int_M (u_\varepsilon - c_\varepsilon)^2 |\nabla \eta|^2 dv_g + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g \right)^2.$$

From (15) we deduce that

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant$$
$$\frac{\varepsilon^2}{8} \mu_1(g_\varepsilon) \int_M u_\varepsilon (u_\varepsilon - c_\varepsilon)\eta^2 f_{\alpha,\varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g \right)^2.$$

First case : assume that $\int_M u_\varepsilon (u_\varepsilon - c_\varepsilon)\eta^2 f_{\alpha,\varepsilon}^2 dv_g \geqslant 0$.
The relation (14) implies

$$\int_M (u_\varepsilon - c_\varepsilon)^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant k \int_M u_\varepsilon (u_\varepsilon - c_\varepsilon)\eta^2 f_{\alpha,\varepsilon}^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g \right)^2.$$

A straightforward computation shows that

$$(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant$$
$$(2-k) c_\varepsilon \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g \right)^2. \tag{18}$$

Now note that

$$\begin{aligned}
\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g &= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 dv_g \\
&= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g + \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g \\
&= \int_M u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g \\
&\leqslant \int_{M \setminus B_p(\alpha/2)} u_\varepsilon f_{\alpha,\varepsilon}^2 (\eta^2 - 1) dv_g
\end{aligned}$$

and from the definition of $f_{\alpha,\varepsilon}$ and η and from the fact that u_ε is bounded in L^2, we deduce that

$$\int_M u_\varepsilon f_{\alpha,\varepsilon}^2 \eta^2 dv_g = O(\varepsilon^4).$$

Since c_ε is bounded (18) becomes

$$\begin{aligned}
(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g &\leqslant O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M (u_\varepsilon - c_\varepsilon) \eta f_{\alpha,\varepsilon}^2 dv_g \right)^2 \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g + \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2 \\
&= O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2} \left(\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g - c_\varepsilon \int_M f_{\alpha,\varepsilon}^2 \eta \right)^2 \qquad (19)
\end{aligned}$$

where in the last equality we have used the fact that $\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon dv_g = \frac{1}{\mu_1(g_\varepsilon)} \int_M \Delta u_\varepsilon dv_g = 0$.

Using the same arguments as above we see that $\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon (\eta - 1) dv_g = O(\varepsilon^4)$. Reporting this in (19) we get

$$(1-k) \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 \eta^2 dv_g + c_\varepsilon^2 \int_M f_{\alpha,\varepsilon}^2 \eta^2 dv_g \leqslant O(\varepsilon^4) + \frac{\varepsilon^2}{8} \|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{O(\varepsilon^4)}{\varepsilon^2} \int_M f_{\alpha,\varepsilon}^2 \eta dv_g + \frac{c_\varepsilon^2}{\pi\varepsilon^2} \left(\int_M f_{\alpha,\varepsilon}^2 \eta dv_g \right)^2.$$

Now

$$\begin{aligned}
\int_M f_{\alpha,\varepsilon}^2 \eta dv_g = \int_{B_p(\alpha)} f_{\alpha,\varepsilon}^2 dv_g &= \int_0^{2\pi} \int_0^\alpha \frac{\varepsilon^4 r}{(\varepsilon^2 + r^2)^2} dr d\Theta \\
&= 2\pi\varepsilon^2 \int_0^{\alpha/\varepsilon} \frac{t}{(1+t^2)^2} dt \\
&\leqslant 2\pi\varepsilon^2 \int_0^{+\infty} \frac{t}{(1+t^2)^2} dt \\
&= \pi\varepsilon^2.
\end{aligned}$$

This gives

$$\begin{aligned}(1-k)\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2\eta^2 dv_g + c_\varepsilon^2\int_M f_{\alpha,\varepsilon}^2\eta^2 dv_g &\leqslant O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{c_\varepsilon^2}{\pi\varepsilon^2}\left(\int_M f_{\alpha,\varepsilon}^2\eta dv_g\right)^2\\ &= O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2\int_M f_{\alpha,\varepsilon}^2\eta dv_g.\end{aligned}$$

Finally we have

$$\begin{aligned}(1-k)\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2\eta^2 dv_g &\leqslant O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2\int_M f_{\alpha,\varepsilon}^2(\eta-\eta^2) dv_g\\ &\leqslant O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + c_\varepsilon^2\int_{B_p(\alpha)\setminus B_p(\alpha/2)} f_{\alpha,\varepsilon}^2 dv_g\\ &= O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2. \end{aligned}\tag{20}$$

Second case : Assume that $\int_M u_\varepsilon(u_\varepsilon - c_\varepsilon)\eta^2 f_{\alpha,\varepsilon}^2 dv_g \leqslant 0$.

In this case, we have

$$\begin{aligned}&\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2\eta^2 dv_g - 2c_\varepsilon\int_M u_\varepsilon f_{\alpha,\varepsilon}^2\eta^2 dv_g + c_\varepsilon^2\int_M f_{\alpha,\varepsilon}^2\eta^2 dv_g \leqslant\\ &\qquad O(\varepsilon^4)+\frac{\varepsilon^2}{8}\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \frac{1}{\pi\varepsilon^2}\left(\int_M (u_\varepsilon - c_\varepsilon)\eta f_{\alpha,\varepsilon}^2 dv_g\right)^2\end{aligned}\tag{21}$$

and we conclude as in the previous case.

Then we have proved that

$$\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2\eta^2 dv_g = O(\varepsilon^4+\varepsilon^2\|u_\varepsilon - c_\varepsilon\|_{L^2}^2).$$

To finish the proof, we write

$$\int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 dv_g = \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2\eta^2 dv_g + \int_M u_\varepsilon^2 f_{\alpha,\varepsilon}^2 (1-\eta^2) dv_g$$

and the last term is $O(\varepsilon^4)$ which completes the proof.

□

Proof. of Relation (13). First we apply the lemma 5.4 to $c_\varepsilon = c_0$ and we see that $c_0 \neq 0$. Indeed, let us compute the L^2-norm of the gradient of u_ε.

$$\begin{aligned}\int_M |\nabla u_\varepsilon|^2 dv_g = \int_M (\Delta u_\varepsilon)u_\varepsilon dv_g &\leqslant \frac{8k}{\varepsilon^2}\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g\\ &= \frac{8k}{\varepsilon^2}O(\varepsilon^2\|u_\varepsilon - c_0\|_{L^2}^2+\varepsilon^4)\\ &= o(1).\end{aligned}$$

Then we deduce that up to a subsequence

$$\int_M |\nabla u_\varepsilon|^2 dv_g \longrightarrow 0.$$

But we have chosen u_ε so that $\|u_\varepsilon\|_{H_1^2} = 1$. Then $\|u_\varepsilon\|_{L^2} \longrightarrow 1$ and $c_0 \neq 0$.

Now let us consider $\overline{u}_\varepsilon = \frac{1}{\mathrm{Vol}(M)} \int_M u_\varepsilon dv_g$ and $a_\varepsilon = \|u_\varepsilon - \overline{u}_\varepsilon\|_{H_1^2}$. Then $u_\varepsilon \longrightarrow c_0$ and $a_\varepsilon \longrightarrow 0$. It follows that the function $v_\varepsilon = \frac{u_\varepsilon - \overline{u}_\varepsilon}{a_\varepsilon}$ satisfies $\|v_\varepsilon\|_{H_1^2} = 1$ and there exists $v \in H_1^2$ so that $v_\varepsilon \longrightarrow v$ weakly in H_1^2 and strongly in L^2.

To prove (13) we will consider two cases.

First case : Assume that up to a subsequence $a_\varepsilon = O(\varepsilon)$.

We have

$$\begin{aligned}
\int_M (\Delta u_\varepsilon)^2 dv_g &= \mu_1(g_\varepsilon)^2 \int_M f_{\alpha,\varepsilon}^4 u_\varepsilon^2 dv_g \\
&\leqslant \mu_1(g_\varepsilon)^2 \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g \\
&\leqslant \frac{64k}{\varepsilon^4} O(\varepsilon^2 \|u_\varepsilon - \overline{u}_\varepsilon\|_{L^2}^2 + \varepsilon^4) \\
&\leqslant \frac{64k}{\varepsilon^4} O(\varepsilon^2 a_\varepsilon^2 + \varepsilon^4) \\
&\leqslant M.
\end{aligned}$$

Then $\|\Delta u_\varepsilon\|_{L^2}$, $\|\nabla u_\varepsilon\|_{L^2}$ and $\|u_\varepsilon\|_{L^2}$ are bounded. It well known that the norms

$$\| v \| := \| \Delta v \|_{L^2} + \| \nabla v \|_{L^2} + \| v \|_{L^2}$$

and $\| v \|_{H_2^2}$ are equivalent (it is a direct consequence of Bochner formula). Hence, this implies that $(u_\varepsilon)_\varepsilon$ is bounded in H_2^2 which is embedded in C^0. Then $u_\varepsilon \longrightarrow c_0$ uniformily up to a subsequence. Since $c_0 \neq 0$ it follows that for ε small enough u_ε has a constant sign, which is not possible because u_ε is an eigenfunction in the metric g_ε.

Second case : Assume that $\varepsilon = a_\varepsilon o(1)$. In this case we have the

Lemma 5.5. *$v_\varepsilon \longrightarrow c_1$ in H_1^2 where c_1 is a constant.*

Proof. The proof is similar to this of lemma 5.3. Indeed we consider $\varphi \in C^\infty(M)$ and the function η_ρ defined in this previous proof. Then

$$\int_M \langle \nabla v, \nabla \varphi \rangle = \int_M \langle \nabla v, \nabla(\eta_\rho \varphi) \rangle dv_g + \int_M \langle \nabla v, \nabla((1-\eta_\rho)\varphi) \rangle dv_g.$$

By the same arguments we have $\int_M \langle \nabla v, \nabla(\eta_\rho \varphi) \rangle dv_g \longrightarrow 0$ when $\rho \longrightarrow 0$. Moreover

$$\begin{aligned}
\left| \int_M \langle \nabla v, \nabla\left((1-\eta_\rho)\varphi\right) \rangle dv_g \right| &= \lim_{\varepsilon \longrightarrow 0} \left| \int_M \langle \nabla v_\varepsilon, \nabla\left((1-\eta_\rho)\varphi\right) \rangle dv_g \right| \\
&= \lim_{\varepsilon \longrightarrow 0} \left| \int_M (\Delta v_\varepsilon)(1-\eta_\rho)\varphi dv_g \right| \\
&= \lim_{\varepsilon \longrightarrow 0} \left| \frac{\mu_1(g_\varepsilon)}{a_\varepsilon} \int_M f_{\alpha,\varepsilon}^2 v_\varepsilon (1-\eta_\rho)\varphi dv_g \right|.
\end{aligned}$$

Now $\left|\frac{\mu_1(g_\varepsilon)}{a_\varepsilon}\int_M f_{\alpha,\varepsilon}^2 v_\varepsilon(1-\eta_\rho)\varphi dv_g\right| \leqslant \frac{8k}{a_\varepsilon \varepsilon^2} C\varepsilon^4 \|v_\varepsilon\|_{L^2}\left(\int_M (1-\eta_\rho)\varphi^2 dv_g\right)^{1/2}$. Since $\varepsilon = a_\varepsilon o(1)$, we deduce that $\left|\int_M \langle \nabla v, \nabla\left((1-\eta_\rho)\varphi\right)\rangle dv_g\right| = 0$ and then $\int_M \langle \nabla v, \nabla \varphi\rangle = 0$. Therefore $\Delta v = 0$ in sense of distributions and $v = c_1$ on M.

□

Now let $c_\varepsilon = \overline{u}_\varepsilon + a_\varepsilon c_1$. Then $c_\varepsilon \longrightarrow c_0$. We denotes by $\mu(g)$ the smallest positive eigenvalue of the Laplacian with respect to the metric g. From the definition of a_ε and the definition of $\mu(g)$, we have

$$\begin{aligned} a_\varepsilon^2 \leqslant 2\left(\int_M |\nabla u_\varepsilon|^2 dv_g + \int_M (u_\varepsilon - \overline{u}_\varepsilon)^2 dv_g\right) &\leqslant 2\left(1+\frac{1}{\mu(g)}\right)\int_M |\nabla u_\varepsilon|^2 dv_g \\ &= 2\left(1+\frac{1}{\mu(g)}\right)\int_M \Delta u_\varepsilon u_\varepsilon dv_g \\ &= 2\left(1+\frac{1}{\mu(g)}\right)\mu_1(g_\varepsilon)\int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g. \end{aligned} \tag{22}$$

Applying lemma 5.4 we get

$$\begin{aligned} \int_M f_{\alpha,\varepsilon}^2 u_\varepsilon^2 dv_g &= O(\varepsilon^2\|u_\varepsilon - c_\varepsilon\|_{L^2}^2 + \varepsilon^4) \\ &= O(\varepsilon^2\|u_\varepsilon - \overline{u}_\varepsilon - a_\varepsilon c_1\|_{L^2}^2 + \varepsilon^4) \\ &= O\left(a_\varepsilon^2\varepsilon^2\left\|\frac{u_\varepsilon - \overline{u}_\varepsilon}{a_\varepsilon} - c_1\right\|_{L^2}^2 + \varepsilon^4\right) \\ &= O(a_\varepsilon^2\varepsilon^2\|v_\varepsilon - c_1\|_{L^2}^2 + \varepsilon^4) \\ &= O(\varepsilon^4) + o(a_\varepsilon^2\varepsilon^2). \end{aligned}$$

Now reporting this in (22) with the estimate (14) we find

$$\begin{aligned} a_\varepsilon^2 &\leqslant C\frac{8k}{\varepsilon^2}(O(\varepsilon^4) + o(a_\varepsilon^2\varepsilon^2)) \\ &= O(\varepsilon^2) + a_\varepsilon^2 o(1). \end{aligned}$$

But $\varepsilon = a_\varepsilon o(1)$. Then $a_\varepsilon^2 \leqslant C a_\varepsilon^2 o(1)$ and for ε small enough $a_\varepsilon = 0$ and u_ε is a constant which is impossible.

□

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